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Discussion Papers
Department of Economics
University of Copenhagen

No. 07-02

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ISSN: 1601-2461 (online)

Decreasing serial cost sharing: an axiomatic characterization

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January 2007

Abstract

The increasing serial cost sharing rule of Moulin and Shenker [Econometrica 60 (1992) 1009] and the decreasing serial rule of de Frutos [Journal of Economic Theory 79 (1998) 245] have attracted attention due to their intuitive appeal and striking incentive properties. An axiomatic characterization of the increasing serial rule was provided by Moulin and Shenker [Journal of Economic Theory 64 (1994) 178]. This paper gives an axiomatic characterization of the decreasing serial rule.

Keywords: Serial cost sharing, Cost allocation, Axiomatic characterization.

JEL classification: D23, D71, L24.

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1 Introduction

We consider cost sharing problems with a single good. Each agent in a group demands a non-negative amount and the total cost of supplying the aggregate demand is specified by a cost function. A *cost sharing rule* assigns shares of the total cost to each agent.

Two cost sharing rules have received considerable attention due to their intuitive appeal and striking incentive properties; the increasing and decreasing serial rules (see, e.g., Moulin and Shenker 1992, 1994, Moulin 1996, de Frutos 1998 and Hougaard and Thorlund-Petersen 2000, 2001).

According to the increasing serial rule (originally called serial cost sharing in Moulin and Shenker 1992) each agent's cost share is determined by his own demand and the demands of those agents who have the same or a smaller demand. This rule seems particularly appropriate in situations where the cost functions are convex as argued in Hougaard and Thorlund-Petersen (2001). Increasing serial cost sharing is characterized in Moulin and Shenker (1992) and Moulin (1996) in terms of the strategic properties of the induced cost sharing game. Moulin and Shenker (1994) provides an axiomatic characterization.

According to the decreasing serial rule (de Frutos 1998) each agent's cost share is determined by his own demand and the demands of those other agents who have the same or a *larger* demand. This rule seems particularly appropriate in a situation where the cost function is *concave*; in fact it only guarantees non-negative cost shares in this case. In de Frutos (1998) decreasing serial cost sharing is characterized in terms of the strategic properties of the induced cost sharing game. However, as noted by Moulin (2002), the decreasing cost sharing rule has not yet received an axiomatic characterization.

This paper provides such an axiomatic characterization of the decreasing serial rule. To facilitate meaningful comparisons, and in an attempt to develop a unified approach, it has been our aim to obtain an axiomatic characterization which parallels that given by Moulin and Shenker (1994) where the increasing serial rule is characterized by the axioms of Separable Costs and Additivity (see Section 3) and a sort of a consistency axiom called Free Lunch which is discussed in Section 4. But while there are clear similarities between the increasing and decreasing serial rules (and their axiomatic characterizations) there are also some important differences. Indeed, for reasons that we shall explain later, for the decreasing serial rule there appears to be no direct counterpart to the Free Lunch axiom of the Moulin-Shenker characterization.

2 Definitions

Let $C : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing *cost function* with $C(0) = 0$, and let $N = \{1, ..., n\}$ be a finite set of agents. Demands are given by a vector $q \in \mathbb{R}^N_+$, where q_i denotes agent i's demand. We assume that agents are numbered such that $q_1 \leq \cdots \leq q_n$.

The triple (N, C, q) constitutes a *cost sharing problem.* Given (N, C, q), a *cost allocation* is a vector x in $\mathbb{R}^N$ such that $\sum_{i=1}^n x_i = C(\sum_{i=1}^n q_i)$. A *cost sharing rule* is a function ϕ that to each cost sharing problem (N, C, q)

assigns a cost allocation $x = \phi(N, C, q)$, where $x_i = \phi_i(N, C, q)$ is the cost share of agent $i \in N$.

Given a cost sharing problem (N, C, q), we define $r_0 = 0, r_1 = nq_1, r_2 = q_1 + (n-1)q_2$ and $r_i = q_1 + ... + q_{i-1} + (n+1-i)q_i$ for $i = 3, .., n$, and $s_0 = 0, s_1 = nq_n, s_2 = q_n + (n-1)q_{n-1}$ and $s_i = q_n + ... + q_{n+2-i} + (n+1-i)q_{n+1-i}$, for $i = 3, ..., n$. Note that $r_1 \leq \cdots \leq r_n = q_1 + ... + q_n = s_n \leq \cdots \leq s_1$. The *increasing serial rule* (Moulin and Shenker 1992, 1994) is defined by

$$\phi_i^I(C, N, q) = \sum_{k=1}^{i} \frac{C(r_k) - C(r_{k-1})}{n+1-k}, \quad i = 1, \ldots, n. \tag{1}$$

The *decreasing serial rule* (de Frutos 1998) is defined by

$$\phi_i^D(C, N, q) = \sum_{k=1}^{n+1-i} \frac{C(s_k) - C(s_{k-1})}{n+1-k}, \quad i = 1, \ldots, n. \tag{2}$$

We say that a cost sharing rule ϕ is *continuous* if whenever C_k converges pointwise to C, then $\phi_i(N, C_k, q)$ converges to $\phi_i(N, C, q)$, for all N, i, and q. A cost sharing rule ϕ is *order-preserving* if $\phi_1(N, C, q) \leq \cdots \leq \phi_n(N, C, q)$ whenever $q_1 \leq \cdots \leq q_n$. Continuity and order-preservation are standard regularity conditions, and in the following we restrict attention to cost sharing rules satisfying both conditions.

3 An axiomatic characterization

Our characterization combines two well-known properties with a third property that is new but in a sense related to the Free Lunch axiom in Moulin and Shenker (1994).

Let C^α denote the linear function for which $C(z) = \alpha z$ for all $z \geq 0$.

Axiom 1 (Separable Costs). If $C = C^\alpha$ then $\phi_i(N, C^\alpha, q) = \alpha q_i$ for all i.

Separable Costs states that in case of a linear cost function all agents pay the constant average cost for all units demanded. Since the increasing and decreasing serial rules coincide on linear cost functions both rules satisfy Separable Costs.

Axiom 2 (Additivity). $\phi(N, C_1 + C_2, q) = \phi(N, C_1, q) + \phi(N, C_2, q)$ for any cost functions C_1, C_2.

Additivity states that the cost shares are independent of whether any two cost sharing problems are resolved together or separately. It is satisfied by both the increasing and decreasing serial rule.

For the third axiom we make use of the following definitions: For $z \in \mathbb{R}$, define $(z)_+ = \max\{0, z\}$. For $t \in \mathbb{R}_+$, let $\Delta_t(z) = \min\{z, t\}$ be the *plateau* cost function where total cost is equal to total demand until a plateau level t is reached and the total cost then remains fixed. For $S \subset N$, q^S is the projection of q on $\mathbb{R}^S$.

Axiom 3 (Plateau Cost). If $C = \Delta_t$ and $nq_n \geq t$, then $\phi_n(N, \Delta_t, q) = t/n$ and $\phi_i(N, \Delta_t, q) = \phi_i(N \backslash \{n\}, \Delta_{(t-q_n)_+}, q^{N \backslash \{n\}}) + \frac{q_n - (q_n - t)_+ - t/n}{n-1}$ for $i \neq n$.

This axiom states that if C is a plateau cost function with plateau level t, the agent with the highest demand pays an equal share of the plateau cost t if the total demand would exceed t in the case that all agents demanded the same quantity as the agent with the highest demand. This might seem reasonable considering that average costs are decreasing and the group as a whole benefits from a large total demand. Now, having settled the cost share of the agent with the highest demand, this agent may now be removed from the set of agents and the cost shares of the remaining agents can be specified by imposing the same cost sharing rule on an adjusted cost function and adding a constant. The plateau level of the cost function is adjusted for the size of the highest demand, and a constant is added in order to ensure budget balance, i.e. to adjust for differences between the size of the demand and the size of the cost share of the agent with the highest demand. Note that if the demand of agent n alone is above the plateau t then all agents share the fixed cost equally. The axiom is silent in cases where $nq_n < t$.

It is easy to see that Plateau Cost is violated by increasing serial cost sharing: For example, let $C = \Delta_t$ and let $q = (0, t)$. Then according to the increasing serial rule $\phi_1^I(\Delta_t, q) = 0$ and $\phi_2^I(\Delta_t, 0) = t$. Note, that according to the decreasing serial rule $\phi_1^D(\Delta_t, q) = \phi_2^D(\Delta_t, q) = t/2$ in line with Plateau Cost. Actually, the plateau cost functions can be used to give us extreme examples of how the rules differ on concave cost functions: the increasing rule yields the most unequal allocation while the decreasing rule yields the most equal allocation (see Lemma 4 in Hougaard and Thorlund-Petersen, 2001, for a precise statement). Lemma 1 in the appendix specifies how the decreasing serial rule generally works on plateau cost functions.

We shall now state the main result.

Theorem 1 *A continuous and order-preserving cost sharing rule ϕ satisfies Axioms 1-3 if and only if it is the decreasing serial cost sharing rule ϕ^D.*

The proof is provided in the appendix. Note that the axioms are independent: The Equal split rule, $\phi_i(C, N, q) = C(\sum_{j=1}^n q_j)/n$ for all $i \in N$, satisfies Axioms 2 and 3 but not 1 (Separable Costs); Mixed serial cost sharing as defined in Hougaard and Thorlund-Petersen (2001) satisfies Axioms 1 and 3 but not 2 (Additivity); and Increasing serial cost sharing satisfies Axioms 1 and 2 but not 3 (Plateau Cost).

4 Comments

Comparing the two axiomatizations of increasing and decreasing serial cost sharing the difference lies in the Free Lunch axiom of Moulin and Shenker (1994) versus the Plateau Cost axiom of the present characterization; the other axioms are the same. Roughly speaking, the Free Lunch axiom states that if $C(nq_i) = 0$ then agent i pays nothing - gets a 'free lunch', and the cost shares of the other agents are unaffected by removing agent i from the cost sharing problem and adjusting the cost function accordingly (such that the cost function C is replaced by the function $\overline{C}$ defined by $\overline{C}(z) = \max\{C(z+q_i)-\phi_i(N, C, q), 0\}$ if $z > 0$ and $\overline{C}(0) = 0$). Hence, Free Lunch is a limited consistency property relying on a suitable definition of the 'reduced' cost sharing problem just like the Plateau Cost axiom.

Since any non-decreasing and convex function can be approximated by a weighted sum of slant functions $\Lambda_t(z) = (z - t)_+$, the proof in Moulin and Shenker (1994) uses only the Free Lunch axiom in relation to slant functions. Therefore a direct mirror-image of the Moulin-Shenker proof would involve an axiom related to (affine) two-part tariffs. But since non-decreasing and concave functions generally cannot be approximated by a weighted sum of two-part tariffs we use the closest such relative - the angle functions - or rather their extreme form, the plateau cost functions Δ_t. It makes a difference whether we consider two-part tariffs or plateau cost functions and basically it is this difference that is reflected in our formulation of Plateau Cost versus the formulation of Free Lunch.

Following common convention, we have assumed that $C(0) = 0$. This assumption is restrictive, since it rules out the cost sharing problems involving a fixed cost. The assumption plays a role for an axiomatic characterization of the increasing serial rule, because if $C(0)$ were allowed to be positive the increasing serial rule would actually fail to be continuous. In contrast, continuity of decreasing serial rule does not require that $C(0) = 0$. Indeed, it is easy to verify that our definitions and characterization applies also to the domain of cost function $C : \mathbb{R}_+ \to \mathbb{R}_+$.

As mentioned previously, the increasing (decreasing) serial rule seems particularly appropriate in a situation where the cost function is convex (concave). However, neither Moulin and Shenker's axiomatic characterization of the increasing serial rule nor our characterization of the decreasing serial rule applies to the more restrictive domains of convex (concave) cost functions. The reason is that both characterizations make use of the fact that a plateau cost function can be written as the difference between a linear cost function and a slant function. Hence, both concave plateau function and convex slant functions must be contained in the relevant domain of cost function. Axiomatic characterizations on either concave or convex domains is an interesting topic for future research.

A Appendix: Proof of result

For the proof of Theorem 1, we first specifies the decreasing serial rule for the case of plateau cost functions Δ_t.

Lemma 1 *Let $t > 0$, and let (N, Δ_t, q) be a cost sharing problem. If $s_1 < t$, then $\phi_j^D(N, \Delta_t, q) = q_j$ for $j = 1, \ldots, n$. If $s_n \geq t$ then $\phi_j^D(\Delta_t, q) = \frac{t}{n}$ for all $j = 1, \ldots, n$. If $s_n \leq t$ and $s_1 \geq t$, let i be the smallest index for which $s_{n+1-i} \geq t$. Then*

$$\phi_j^D(N, \Delta_t, q) = \frac{t}{n},$$

$j = i, \ldots, n$, and

$$\phi_j^D(N, \Delta_t, q) = q_j + \frac{\sum_{k=i}^n q_k - (n+1-i)t/n}{i-1},$$

$j = 1, \ldots, i-1$.

Proof: If $s_1 < t$ then from (2) it follows readily that $\phi_j^D(N, \Delta_t, q) = q_j$ for $j = 1, \ldots, n$. Suppose therefore that $s_1 \geq t$, and let i be the smallest index for which $s_{n+1-i} \geq t$. If $s_n \geq t$ then it follows from (2) that $\phi_j^D(N, \Delta_t, q) = \frac{t}{n}$ for all $j = 1, \ldots, n$ (since $s_k \geq t$ for $k = 1, \ldots, n$).

If $s_n < t$, then, for $j = i, \ldots, n$ we have

$$\phi_j^D(N, \Delta_t, q) = \sum_{k=1}^{n+1-j} \frac{\Delta_t(s_k) - \Delta_t(s_{k-1})}{n+1-k} = \frac{t}{n}$$

since $\Delta_t(s_k) = t$ for $k = 1, \ldots, n+1-i$. Moreover, for agent $i-1$ by (2) we have

$$
\begin{aligned}
\phi^D_{i-1}(N,\Delta_t,q) &= \frac{t}{n}+\frac{s_{n+2-i}-t}{i-1} \\
&= -\frac{(n+1-i)t}{n(i-1)}+\frac{\sum_{k=i}^n q_k}{i-1}+q_{i-1} \\
&= \frac{\sum_{k=i}^n q_k-(n+1-i)t/n}{i-1}+q_{i-1}.
\end{aligned}
$$

For $j=1,\ldots,i-2$ by (2) we have

$$
\begin{aligned}
\phi^D_{j}(N,\Delta_t,q) &= \frac{t}{n}+\frac{s_{n+2-i}-t}{i-1}+\sum_{k=n+3-i}^{n+1-j}\frac{s_k-s_{k-1}}{n+1-k} \\
&= -\frac{(n+1-i)t}{n(i-1)}+\frac{\sum_{k=i}^n q_k}{i-1}+q_j \\
&= \frac{\sum_{k=i}^n q_k-(n+1-i)t/n}{i-1}+q_j.
\end{aligned}
$$

since $\Delta_t(s_k)=s_k$ for $k=n+2-i,\ldots,n+1-j$. □

We are now ready to prove our main result.

Proof of Theorem 1: First we show, in five steps, that the decreasing serial rule is the *only* rule (if any) that can be consistent with Axioms 1-3. The sixth and last step verifies that the decreasing serial rule is, in fact, consistent with the axioms.

Step 1: By Axioms 1-3 we get that $\phi_n(N,\Lambda_{nq_n},q)=\phi_n(N,C^1,q)-\phi_n(N,\Delta_{nq_n},q)=q_n-(nq_n)/n=0$. Define $\Lambda_t(z)=(z-t)_+$. By order-preservation, $\phi_i(N,\Lambda_{nq_n},q)\leq 0$, for $i=1,\ldots,n-1$. Now, since $\sum_{i=1}^n\phi_i(N,\Lambda_{nq_n},q)=0$ we have $\phi_i(N,\Lambda_{nq_n},q)=0$, for all i. Thus, by Axioms 1 and 2

$$\phi_i(N,\Delta_{nq_n},q)=\phi_i(N,C^1,q)-\phi_i(N,\Lambda_{nq_n},q)=q_i-0=q_i,$$

for all i.

Now, add a new agent $n+1$ with $q_{n+1} \geq q_n$ and let $\bar{N} = N \cup \{n+1\}$. By Axiom 3, $\phi_{n+1}(\bar{N}, \Delta_{(n+1)q_{n+1}}, (q, q_{n+1})) = q_{n+1}$ and by the above argument

$$\phi_i(\bar{N}, \Delta_{(n+1)q_{n+1}}, (q, q_{n+1})) = q_i \quad \text{for all} \quad i \leq n.$$

Using Axiom 3 we remove the new agent again and obtain

$$\phi_i(\bar{N}, \Delta_{(n+1)q_{n+1}}, (q, q_{n+1})) = \phi_i(N, \Delta_{nq_{n+1}}, q) + 0 = q_i,$$

for $i \leq n$. This, in turn, determines $\phi_i(N, \Delta_t, q)$ for large t:

$$\phi_i(N, \Delta_t, q) = q_i,$$

for all i and all $t \geq nq_n = s_1$. Thus, for $t \geq s_1$, $\phi(N, \Delta_t, q) = \phi^D(N, \Delta_t, q)$ by Lemma 1. In the remainder of the proof we assume that $t < s_1$.

Step 2: By Axiom 3 we have $\phi_n(N, \Delta_t, q) = t/n$ and for $i \neq n$

$$\phi_i(N, \Delta_t, q) = \phi_i(N \backslash \{n\}, \Delta_{(t-q_n)_+}, q^{N \backslash \{n\}}) + \frac{q_n - (q_n - t)_+ - t/n}{n-1}.$$

If $q_n \geq t$ then by Axiom 3 (and using continuity for the special case $q_n = t$) we have $\phi_i(N, \Delta_t, q) = t/n$ for all $i \neq n$. Thus, for $t < s_1$ and $q_n \geq t$, $\phi(N, \Delta_t, q) = \phi^D(N, \Delta_t, q)$. In the remainder of the proof we assume that $q_n < t$.

Step 3: By Axiom 3 we have

$$\phi_i(N, \Delta_t, q) = \phi_i(N \backslash \{n\}, \Delta_{t-q_n}, q^{N \backslash \{n\}}) + \frac{q_n - t/n}{n-1}, \text{ for } i \neq n.$$

Consider the agent with the second largest demand q_{n-1}. If $s_2 \le t$ then (by the arguments above) we have

$$\phi_i(N\backslash\{n\}, \Delta_{t-q_n}, q^{N\backslash\{n\}}) = q_i,$$

for all $i = 1, \ldots, n-1$, and consequently $\phi_i(N, \Delta_t, q) = q_i + \frac{q_n - t/n}{n-1}$ for $i \neq n$. Hence, for $s_2 \le t < s_1$ and $t > q_n$, $\phi(N, \Delta_t, q) = \phi^D(N, \Delta_t, q)$. In the remainder of the proof we assume that $s_2 > t$.

By Axiom 3, $\phi_{n-1}(N\backslash\{n\}, \Delta_{t-q_n}, q^{N\backslash\{n\}}) = \frac{t-q_n}{n-1}$ which implies $\phi_{n-1}(N, \Delta_t, q) = \frac{t-q_n}{n-1} + \frac{q_n - t/n}{n-1} = t/n$, and thus for $q_n < t < s_2$ we have $\phi_{n-1}(N, \Delta_t, q) = \phi^D_{n-1}(N, \Delta_t, q)$ by Lemma 1.

Step 4: Now, consider an arbitrary $i \le n-2$, and suppose that $\phi_j(N, \Delta_t, q) = \phi^D_j(N, \Delta_t, q)$ for all $j = i+1, ..., n$. We will show that $\phi_i(N, \Delta_t, q) = \phi^D_i(N, \Delta_t, q)$. For this, we consider two separate cases: i) $t - \sum_{j=i+1}^n q_j \ge 0$, and ii) $t - \sum_{j=i+1}^n q_j < 0$.

Case i). Repeated use of Axiom 3 gives us

$$\begin{aligned}
\phi_i(N, \Delta_t, q) &= \phi_i(N\backslash\{i+1, ..., n\}, \Delta_{t-\sum_{j=i+1}^n q_j}, q^{N\backslash\{i+1,...,n\}}) \\
&\quad + \frac{q_n - t/n}{n-1} + \frac{q_{n-1} - \frac{t-q_n}{n-1}}{n-2} + ... + \frac{q_{i+1} - \frac{t-\sum_{k=i+2}^n q_k}{i+1}}{i} \\
&= \phi_i(N\backslash\{i+1, ..., n\}, \Delta_{t-\sum_{j=i+1}^n q_j}, q^{N\backslash\{i+1,...,n\}}) \\
&\quad + \frac{\sum_{k=i+1}^n q_k - (n-i)t/n}{i}.
\end{aligned}$$

If $s_{n+1-i} \ge t$, then by Axiom 3 we get

$$\phi_i(N\backslash\{i+1, ..., n\}, \Delta_{(t-\sum_{j=i+1}^n q_j)_+}, q^{N\backslash\{i+1,...,n\}}) = \frac{t - \sum_{k=i+1}^n q_k}{i},$$

hence

$$\begin{aligned}\phi_i(N,\Delta_t,q) &= \frac{\sum_{k=i+1}^{n} q_k - (n-i)t/n}{i} + \frac{t - \sum_{k=i+1}^{n} q_k}{i} \\ &= \frac{t}{n}.\end{aligned}$$

If $s_{n+1-i} < t$ then by Step 1 we have

$$\phi_i(N\backslash\{i+1,...,n\}, \Delta_{t-\sum_{j=i+1}^{n} q_j}, q^{N\backslash\{i+1,...,n\}}) = q_i$$

and consequently

$$\phi_i(N,\Delta_t,q) = q_i + \frac{\sum_{k=i+1}^{n} q_k - (n-i)t/n}{i}.$$

Using Lemma 1, we can therefore conclude that $\phi_i(\Delta_t,q) = \phi_i^D(\Delta_t,q)$ if $t - \sum_{j=i+1}^{n} q_j \geq 0$ and $t < s_2$.

Case ii). By Lemma 1, we have $\phi_j^D(N,\Delta_t,q) = \frac{t}{n}$ for $j = i+1,...,n$. Moreover, by Axiom 3 and budget balance (i.e. $\sum_{i=1}^{n} \phi_i(N,\Delta_t) = t$) we get $\phi_j(N,\Delta_t,q) = \frac{t}{n}$ for all $j < i+1$. Using Lemma 1 we conclude that $\phi_i(N,\Delta_t,q) = \phi_i^D(N,\Delta_t,q)$ if $t - \sum_{j=i+1}^{n} q_j < 0$. (Note that $t - \sum_{j=i+1}^{n} q_j < 0$ implies $t < s_2$).

We conclude that $\phi(N,\Delta_t) = \phi^D(N,\Delta_t)$.

Step 5: By Axiom 1 and 2, and Lemma 1 in Moulin and Shenker (1994), we have that $\phi(C,q) = \phi^D(C,q)$ for any function C that can be written as the difference between two non-decreasing convex functions, and $C(0) = 0$. By continuity, Axiom 2, and Remark 2 in Moulin and Shenker (1994) we conclude that $\phi(N,C,q) = \phi^D(N,C,q)$ for an arbitrary non-decreasing function C.

Step 6: Lastly, we need to show the decreasing serial cost sharing rule is, in fact, consistent with Axioms 1-3 . It is well-known (or readily verified) that ϕ^D satisfies Axioms 1 and 2, so we focus on Axiom 3. Let $s_1 \geq t$ then according to (2) we get

$$\phi_n^D(N, \Delta_t, q) = \frac{\Delta_t(s_1)}{n} = \frac{t}{n},$$

hence the decreasing serial cost sharing rule satisfies the first part of Axiom 3. Now, if $q_n \geq t$ then $s_k \geq t$ for all $k = 1, \ldots, n$ and consequently $\phi_i^D(N, \Delta_t, q) = t/n$ for all i satisfying Axiom 3. Hence, let $q_n < t$. For $j = n-1$ we have

$$\begin{aligned}
\phi_{n-1}^D(N, \Delta_t, q) &= \frac{\Delta_t(s_2)}{n-1} - \frac{t}{n(n-1)} \\
&= \frac{\Delta_{t-q_n}(s_2 - q_n)}{n-1} + \frac{q_n}{n-1} - \frac{t}{n(n-1)} \\
&= \phi_{n-1}^D(N \backslash \{n\}, \Delta_{t-q_n}, q^{N \backslash \{n\}}) + \frac{q_n - t/n}{n-1}.
\end{aligned}$$

Now consider an arbitrary $j \in \{1, \ldots, n-2\}$. Then

$$\begin{aligned}
\phi_j^D(N, \Delta_t, q) &= \sum_{k=1}^{n+1-j} \frac{\Delta_t(s_k) - \Delta_t(s_{k-1})}{n+1-k} \\
&= \frac{t}{n} + \frac{\Delta_t(s_2) - t}{n-1} + \sum_{k=3}^{n+1-j} \frac{\Delta_t(s_k) - \Delta_t(s_{k-1})}{n+1-k} \\
&= -\frac{t}{n(n-1)} + \frac{\Delta_t(s_2)}{n-1} + \sum_{k=3}^{n+1-j} \frac{\Delta_t(s_k) - \Delta_t(s_{k-1})}{n+1-k} \\
&= -\frac{t}{n(n-1)} + \frac{q_n}{n-1} + \frac{\Delta_{t-q_n}(s_2 - q_n)}{n-1} \\
&\quad + \sum_{k=3}^{n+1-j} \frac{\Delta_{t-q_n}(s_k - q_n) - \Delta_{t-q_n}(s_{k-1} - q_n)}{n+1-k} \\
&= \frac{q_n - t/n}{n-1} + \phi_j^D(N \backslash \{n\}, \Delta_{t-q_n}, q^{N \backslash \{n\}}),
\end{aligned}$$

which shows that ϕ^D also satisfies Axiom 3. $\square$

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